

ON THE POINTWISE ERGODIC BEHAVIOUR OF TRANSFORMATIONS PRESERVING INFINITE MEASURES

BY
JON AARONSON

ABSTRACT

We consider situations in which the asymptotic type of a measure preserving transformation manifests itself in a pointwise manner.

§0. Introduction

In [2], we considered rationally ergodic measure preserving transformations, and we found that the asymptotic behaviour of such transformations depended rather heavily on their asymptotic types. Here we consider pointwise manifestations of these asymptotic types.

Let (X, \mathcal{B}, μ, T) be a conservative measure preserving transformation. We will be considering properties of the form $\exists a_n > 0$ s.t.

$$(0.1) \quad \frac{1}{a_n} \sum_{k=0}^{n-1} f \circ T^k \rightarrow \int_X f d\mu \quad \forall f \in L' \quad \text{in some "pointwise sense"}.$$

If $\mu(X) < \infty$, then the Birkhoff ergodic theorem states that the convergence (0.1) occurs a.e. for $a_n = n/\mu(X)$. It follows from theorem 2 of [1] that if the convergence (0.1) holds a.e. for some constants $a_n > 0$, then $\mu(X) < \infty$ and $a_n \sim n/\mu(x)$. This means that the Hopf ergodic theorem ([7] p. 49)

$$\sum_{k=0}^{n-1} f(T^k x) / \sum_{k=0}^{n-1} (T^k x) \rightarrow \int_X f d\mu / \int_X g d\mu \quad \text{for } \mu\text{-a.e. } x, f, g \in L' \int g \neq 0$$

is the best possible a.e. ergodic theorem for transformations preserving infinite measures.

We will investigate weaker forms of (0.1) which are not entirely eliminated by

Received February 14, 1978

the condition $\mu(X) = \infty$. These properties are *homogeneity* properties, since they will imply that $\sum_{k=0}^n p(T^k x)$ has uniform asymptotic behaviour on X (in the sense that (0.1) holds) for $p \in L^1_+$.

We will need to consider some weakened modes of pointwise convergence of measurable functions.

Let $\{f_n\}$ and f be real valued measurable functions defined on X . We will say that f_n converges to f in μ measure ($f_n \xrightarrow{\mu} f$) if for some (and hence all) finite measures $P \sim \mu$

$$(0.2) \quad \forall \varepsilon > 0 : P(|f_n - f| \geq \varepsilon) \rightarrow 0.$$

We note that $f_n \xrightarrow{\mu} f$ does not necessarily mean that (0.2) holds with $P = \mu$ when $\mu(X) = \infty$. An equivalent way of defining $f_n \xrightarrow{\mu} f$ would be

$$(0.3) \quad \forall n_k \rightarrow \infty \quad \exists m_l = n_{k_l} \rightarrow \infty \quad \text{such that} \quad f_{m_l} \rightarrow f \mu\text{-a.e. on } X.$$

We will say that f_n converges feebly to f ($f_n \rightsquigarrow f$) if

$$(0.4) \quad \forall n_k \rightarrow \infty \quad \exists m_l = n_{k_l} \rightarrow \infty \quad \text{such that} \quad \frac{1}{n} \sum_{i=1}^n f_{m_l} \rightarrow f \mu\text{-a.e. on } X.$$

Feeble convergence (weaker than convergence in measure) can be obtained by

KOMLOS' THEOREM ([4], [9]). If $f_n \in L^1(X)$ and $\sup_{n \geq 1} \int_X |f_n| d\mu < \infty$ then

$$\forall n_k \rightarrow \infty \quad \exists m_l = n_{k_l} \rightarrow \infty \quad \text{and} \quad f \in L^1 \quad \text{such that} \quad f_{m_l} \rightsquigarrow f.$$

In particular, if $f_n \rightarrow f$ weakly in L^1 , then $f_n \rightsquigarrow f$.

In § 1, we prove that if T is rationally ergodic (see [2] for the definition, and [2], [3] for examples) then $\exists a_n > 0$ such that

$$(0.5) \quad \frac{1}{a_n} \sum_{k=0}^{n-1} f \circ T^k \rightsquigarrow \int_X f d\mu \quad \forall f \in L^1.$$

This property, dubbed *weak homogeneity*, is in fact characterised by *slight rational ergodicity* — a very weak form of rational ergodicity.

We say that T is *slightly rationally ergodic* if there are constants $a_n > 0$ such that:

$$\forall n_k \rightarrow \infty \quad \exists m_l = n_{k_l} \rightarrow \infty \quad \text{and} \quad A \in \mathcal{B}, \quad 0 < \mu(A) < \infty, \quad \text{such that}$$

$$(0.6) \quad \frac{1}{a_{m_l}} \sum_{j=0}^{m_l-1} \mu(B \cap T^{-j}C) \rightarrow \mu(B)\mu(C) \quad \forall B, C \in \mathcal{B} \cap A.$$

It is evident that the sequences $\{a_n\}$ in (0.5) and (0.6) are uniquely defined up to asymptotic equality, and it will follow from Theorem 1.1 that they are asymptotically equal to each other (when the same transformation T is involved). No ambiguity arises, therefore, if (as in [2]) we fix one such sequence $a_n = a_n(T)$, and call it a return sequence for T . The collection of all sequences asymptotically proportional to $a_n(T)$ will be called the *asymptotic type of T* and denoted by $\mathcal{A}(T)$.

In the light of Theorem 1.1, we are able to refine, in §3, the results of [2] §2 on the metric invariance of asymptotic type.

Not all c.e.m.p.t.s. are weakly homogeneous. In [6], a c.e.m.p.t. (X, \mathcal{B}, μ, T) is constructed, which commutes with a non-singular transformation $Q: X \rightarrow X$, which does not preserve μ . That is

$$QT = TQ, \quad \mu \circ Q^{-1} \sim \mu \quad \text{and} \quad \mu \circ Q^{-1} \neq \mu.$$

In this situation, as was shown in [6], Q multiplies the measure μ , and can be thought of as “squashing” T into itself. We will, accordingly, call any such transformation T *squashable*. In §2, we derive some consequences of squashability, which show that no squashable transformation is weakly homogeneous.

In §4, we consider the stronger property of *homogeneity*:

$$(0.7) \quad \frac{1}{a_n(T)} \sum_{k=0}^{n-1} f \circ T^k \xrightarrow{\mu} \int_X f d\mu \quad \forall f \in L^1.$$

It turns out that the homogeneity of a transformation is dependent on whether its return time stochastic processes on sets of finite measure satisfy weak laws of large numbers, and that a necessary condition for the homogeneity of a transformation, is that its asymptotic type be regularly varying with index 1. In the case of Markov shifts, this condition is also sufficient. The methods of [1] are applicable to homogeneous transformations, and have analogous consequences.

Part of this work formed part of the author's Ph.D. thesis (as did [1] and [2]), which was written at the Hebrew University of Jerusalem under the supervision of Professor Benjamin Weiss to whom the author is most grateful for many helpful conversations.

§1. Weak homogeneity

We prove:

THEOREM 1.1. *Let (X, \mathcal{B}, μ, T) be a c.e.m.p.t.; then T is weakly homogeneous iff T is slightly rationally ergodic.*

(In this case, the return sequences of T are asymptotically equal).

PROOF. Assume that T is weakly homogeneous. In particular assume that T satisfies the convergence (0.5) with the constants $a_n > 0$ (the return sequence). We first show that

$$(1.1) \quad \lim_{n \rightarrow \infty} \frac{1}{a_n} \sum_{k=0}^{n-1} \mu(B \cap T^{-k}C) \geq \mu(B)\mu(C) \\ \forall B, C \in \mathcal{F} = \{A \in \mathcal{B} : 0 < \mu(A) < \infty\}.$$

To see this, we choose $B, C \in \mathcal{F}$ and let $n_k \rightarrow \infty$ and

$$\frac{1}{a_{n_k}} \sum_{j=0}^{n_k-1} \mu(B \cap T^{-j}C) \rightarrow \alpha \in [0, \infty].$$

By assumption, $\exists m_l = n_{k_l} \rightarrow \infty$ so that

$$(1.2) \quad \frac{1}{n} \sum_{l=1}^n \frac{1}{a_{m_l}} \sum_{j=0}^{m_l-1} f \circ T^j \rightarrow \int_X f d\mu \quad \text{a.e.} \quad \forall f \in L^1.$$

We fix $f = 1_C$, and integrate on B . It follows from Fatou's lemma that $\alpha \geq \mu(B)\mu(C)$.

We now establish the slight rational ergodicity of T .

Let $m_l \rightarrow \infty$ be such that (1.2) is satisfied. Fix $E \in \mathcal{F}$, and choose (using Egorov's theorem) $A \subseteq E$ ($\mu(A) > 0$) so that the convergence (1.2) for $f = 1_E$ is uniform on A .

This implies that the convergence (1.2) for $f = 1_A$ is bounded on A . Hence by the bounded convergence theorem

$$(1.3) \quad \frac{1}{n} \sum_{l=1}^n \frac{1}{a_{m_l}} \sum_{j=0}^{m_l-1} \mu(A \cap T^{-j}A) \rightarrow \mu(A)^2.$$

Now, it is easy to check that if $x_n \geq 0$, $\lim_{n \rightarrow \infty} x_n \geq \alpha$, and $(1/n) \sum_{k=1}^n x_k \rightarrow \alpha$, then $\lim_{n \rightarrow \infty} x_n = \alpha$.

Using this, we derive from (1.1) and (1.3) that $\exists \nu_r = m_{l_r} \rightarrow \infty$ such that

$$(1.4) \quad \frac{1}{a_{\nu_r}} \sum_{j=0}^{\nu_r-1} \mu(A \cap T^{-j}A) \rightarrow \mu(A)^2.$$

We derive (0.6) (for $\{\nu_r\}$ from (1.1) and (1.4) using the method of the proof of proposition 1.1 of [2].

Now assume that T is slightly rationally ergodic. In fact, assume that $\nu_r \rightarrow \infty$ and $A \in \mathcal{F}$ satisfy (0.6). In particular

$$\frac{1}{a_{\nu_i}} \sum_{j=0}^{\nu_i-1} 1_A \circ T^j \rightarrow \mu(A) \quad \text{weakly in } L'(A).$$

By Komlos' theorem $\exists m_i = \nu_i \rightarrow \infty$ such that

$$(1.5) \quad \frac{1}{n} \sum_{i=1}^n \frac{1}{a_{m_i}} \sum_{j=0}^{m_i-1} 1_A \circ T^j \rightarrow \mu(A) \quad \mu\text{-a.e. on } A.$$

The set of points on which the left hand side of (1.5) converges is T -invariant, and, containing A , it must be (almost all) of X , by ergodicity. The Hopf ergodic theorem now establishes (1.2), and hence (0.5). \square

COROLLARY 1.2. *If T is an invertible weakly homogeneous transformation, then so is T^{-1} and $a_n(T^{-1}) \sim a_n(T)$.*

PROOF. Slight rational ergodicity.

§2. Squashable transformations

The following results show that no squashable transformation can be weakly homogeneous.

LEMMA 2.1. *Let (X, \mathcal{B}, μ, T) be a squashable c.e.m.p.t. and let $m_k \rightarrow \infty$ and $d_k > 0$. Then either*

$$\overline{\lim}_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \frac{1}{d_k} \sum_{j=0}^{m_k-1} f \circ T^j = \infty \quad \text{a.e.} \quad \forall f \in L^1_+,$$

or

$$\frac{1}{n} \sum_{k=1}^n \frac{1}{d_k} \sum_{j=0}^{m_k-1} f \circ T^j \rightarrow 0 \quad \text{a.e.} \quad \forall f \in L^1_+.$$

PROOF. Assume $Q : X \rightarrow X$, $\mu \circ Q^{-1} \sim \mu$ and $QT = TQ$. We will show that if the lemma is not true, then $\mu \circ Q^{-1} = \mu$. If the conclusion to the lemma does not hold, we have, using the Hopf ergodic theorem, and the T -super-invariance of the $\overline{\lim}$, that

$$(2.1) \quad \overline{\lim}_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \frac{1}{d_k} \sum_{j=0}^{n_k-1} 1_A \circ T^j = c\mu(A) \quad \text{a.e.} \quad \forall A \in \mathcal{B} \quad \text{where } 0 < c < \infty.$$

Now since $\mu \circ Q^{-1} \sim \mu$, we have that

$$(2.2) \quad \overline{\lim}_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \frac{1}{d_k} \sum_{j=0}^{n_k-1} 1_A \circ T^j \circ Q = c\mu(A) \quad \text{a.e.} \quad \forall A \in \mathcal{B}.$$

And since $QT = TQ$, we have that the left hand side of (2.2) is

$$\overline{\lim}_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \frac{1}{d_k} \sum_{j=0}^{n_k-1} 1_{Q^{-1}A} \circ T^j.$$

Hence, by (2.1) and (2.2), we have $\mu(Q^{-1}A) = \mu(A) \quad \forall A \in \mathcal{B}$. □

THEOREM 2.2. *Let (X, \mathcal{B}, μ, T) be a squashable c.e.m.p.t. and let $A, B \in \mathcal{B}$, $0 < \mu(A)$, $\mu(B) < \infty$, $a_n = \sum_{k=0}^{n-1} \mu(A \cap T^{-k}B)$. Then*

$$\frac{1}{a_n} \sum_{k=0}^{n-1} f \circ T^k \xrightarrow{\mu} 0 \quad \forall f \in L^1.$$

PROOF. Let $n_k \rightarrow \infty$:

$$\int_A \frac{1}{a_{n_k}} \sum_{j=0}^{n_k-1} 1_B \circ T^j d\mu = 1 \quad \forall k \geq 1.$$

By Komlos' theorem, $\exists \phi \in L^1(A)$ and $m_i = n_{k_i} \rightarrow \infty$ such that

$$(2.3) \quad \frac{1}{n} \sum_{i=1}^n \frac{1}{a_{m_i}} \sum_{j=0}^{m_i-1} 1_B \circ T^j \rightarrow \phi \quad \text{a.e. on } A.$$

The set on which this convergence occurs is T -invariant, as is the limit function. Therefore, the convergence (2.3) takes place a.e. on X and the limit function is constant. The constant must be finite since it is integrable on A . But T is squashable, so by Lemma 2.1, the only possible value for this constant is 0.

We now choose a subset C of A ($\mu(C) > 0$) on which the convergence (2.3) is uniform. Integrating, we obtain

$$\frac{1}{n} \sum_{i=1}^n \frac{1}{a_{m_i}} \sum_{j=0}^{m_i-1} \mu(C \cap T^{-j}B) \rightarrow 0.$$

This implies that $\exists \nu_r = m_{i_r} \rightarrow \infty$ such that

$$(2.4) \quad \frac{1}{a_{\nu_r}} \sum_{j=0}^{\nu_r-1} \mu(C \cap T^{-j}B) \rightarrow 0.$$

From here, we see that $\exists p_n = \nu_{r_n} \rightarrow \infty$ such that

$$(2.5) \quad \frac{1}{a_{p_n}} \sum_{j=0}^{p_n-1} 1_B \circ T^j \rightarrow 0 \quad \text{a.e. on } C.$$

As before, this convergence must take place a.e. on X , and using the Hopf ergodic theorem, we have

$$\frac{1}{a_{p_n}} \sum_{j=0}^{p_n-1} f \circ T^j \rightarrow 0 \quad \text{a.e.} \quad \forall f \in L^1. \quad \square$$

§3. Metric invariance of asymptotic type

In this section, we deduce the invariance of the asymptotic type of weakly homogeneous transformations under the metric equivalences defined in [2] §2. We will also obtain that homogeneity properties are invariant under these equivalences, whose definitions we now recall.

Let $(X_i, \mathcal{B}_i, \mu_i, T_i)$ be m.p.t.s., and let $0 < c < \infty$. We will say:

that π is a c -map of T_1 , onto T_2 ($\pi : T_1 \xrightarrow{c} T_2$) if $\pi : X_1 \rightarrow X_2$, $\pi^{-1}\mathcal{B}_2 \subseteq \mathcal{B}_1$, $\pi T_1 = T_2 \pi$ and $\mu_1 \circ \pi^{-1} = c\mu_2$;

that T_2 is a factor of T_1 ($T_1 \rightarrow T_2$) if there is a c -map of T_1 onto T_2 for some $0 < c < \infty$;

that T_1 is similar to T_2 ($T_1 \sim T_2$) if T_1 and T_2 are both factors of the same m.p.t.

The following is immediate from the definitions:

PROPOSITION 3.1. *Let T_1, T_2 be c.e.m.p.t.s. and let $\pi : T_1 \xrightarrow{c} T_2$. Then*

T_1 is homogeneous (weakly homogeneous)

$\Leftrightarrow T_2$ is homogeneous (weakly homogeneous),

and in either case

$$\frac{a_n(T_2)}{a_n(T_1)} \rightarrow c \quad \text{as } n \rightarrow \infty.$$

THEOREM 3.2. *Let T_1, T_2 be c.e.m.p.t.s., and let $T_1 \sim T_2$. Then*

T_1 is homogeneous (weakly homogeneous)

iff T_2 is homogeneous (weakly homogeneous)

and in either case $\mathcal{A}(T_1) = \mathcal{A}(T_2)$.

PROOF. We will show that if T_1 is homogeneous, then so is T_2 and $\mathcal{A}(T_1) = \mathcal{A}(T_2)$. The proof for weak homogeneity is analogous. So we assume that

$$(3.1) \quad \frac{1}{a_{n_k}(T_1)} \sum_{j=0}^{n_k-1} p \circ T_1^j \rightarrow \int_X p d\mu, \quad \mu_1 - \text{a.e.} \quad \forall p \in L^1(X_1)$$

and that $\pi_1 : S \xrightarrow{c} T_1$, $\pi_2 : S \xrightarrow{d} T_1$ where (Y, \mathcal{C}, ν, S) is a m.p.t. necessarily conservative since T_1 is conservative.

The Hopf ergodic theorem applied to S states that

$$(3.2) \quad \sum_{k=0}^{n-1} f(S^k x) / \sum_{k=0}^{n-1} g(S^k x) \rightarrow h \quad \nu\text{-a.e.} \quad \forall f, g \in L^1(Y), \quad g > 0 \quad \text{a.e.}$$

where $h \circ S = h$, $\int_Y |hg| d\nu < \infty$ and $\int_Y hg d\nu = \int_Y f d\nu$.

Combining (3.1) with (3.2) and the fact that T_1 is a c -factor of S yields that

$$(3.3) \quad \frac{1}{a_{n_k}(T_1)} \sum_{j=0}^{n_k-1} f \circ S^j \rightarrow h \quad \nu\text{-a.e.} \quad \forall f \in L^1(Y)$$

where $h \circ S = h$ and $\forall p \in L^1(X_1) : h \cdot p \circ \pi_1 \in L^1(Y)$ and

$$\int_Y hp \circ \pi_1 d\nu = \int_{X_1} p d\mu_1 \int_Y f d\nu.$$

In particular, if h is constant, then

$$(3.4) \quad h = \frac{1}{c} \int_Y f d\nu.$$

Now choose $q \in L^1(X_2)$. We have from (3.3) that

$$(3.5) \quad \frac{1}{a_{n_k}(T_1)} \sum_{j=0}^{n_k-1} q \circ T_2^j \circ \pi_2 = \frac{1}{a_{n_k}(T)} \sum_{j=0}^{n_k-1} q \circ \pi_2 \circ S^j \rightarrow h \quad \nu\text{-a.e.}$$

where $h \circ S = h$. The sequence of functions on the left hand side of (3.5) are all $\pi_2^{-1} \mathcal{B}_2$ -measurable, and hence, so is h . This means that there is a $k : X_2 \rightarrow \mathbf{R}$ such that $h = k \circ \pi_2$. Thus k is a constant, and so is h . By (3.4)

$$h = \frac{1}{c} \int_X q \circ \pi_2 d\nu = \frac{d}{c} \int_{X_2} q d\mu_2.$$

The fact that $\nu \circ \pi_2^{-1} = d\nu_2$ implies that

$$\frac{1}{a_{n_k}(T_1)} \sum_{j=0}^{n_k-1} f \circ T_2^j \rightarrow \frac{d}{c} \int_{X_2} f d\mu_2 \quad \mu_2\text{-a.e.} \quad \forall f \in L^1(X_2). \quad \square$$

§4. Homogeneity

In this section, we consider homogeneity. The homogeneity of a transformation is dependent on whether its return time stochastic processes on sets of finite measure satisfy weak laws of large numbers.

By a *stochastic process*, we mean a quintuple $(\Omega, \mathcal{A}, P, \sigma, \phi)$ where $(\Omega, \mathcal{A}, P, \sigma)$ is a m.p.t., $P(\Omega) = 1$ and ϕ is a real valued measurable function. In this section, we will only be considering ergodic N -valued stochastic processes (i.e. where σ is ergodic and $\phi : \Omega \rightarrow N$). We will say that a stochastic process $(\Omega, \mathcal{A}, P, \sigma, \phi)$

has a *weak law of large numbers* (W.L.L.N.) if there are constants $b_n > 0$ such that

$$(4.1) \quad \frac{1}{b_n} \sum_{k=0}^{n-1} \phi \circ \sigma^k \xrightarrow{P} 1.$$

Now let (X, \mathcal{B}, μ, T) be a c.e.m.p.t. and let $\Omega \in \mathcal{B}$, $0 < \mu(\Omega) < \infty$. We define the *return time stochastic process of T on Ω* , $(\Omega, \mathcal{A}, P, \sigma, \phi)$, by

$$\mathcal{A} = \mathcal{B} \cap \Omega, \quad P(A) = \mu(A \cap \Omega) / \mu(\Omega),$$

$$\phi(x) = \inf \{n \geq 1 : T^n x \in \Omega\} \text{ — the return time function,}$$

$$\sigma x = T^{\phi(x)} x \quad (= T_\Omega x \text{ — the induced transformation}).$$

It follows ([8]) that σ is ergodic and that any (N -valued, ergodic) stochastic process is the return time stochastic process of a c.e.m.p.t. The formula of Kac states that $\int_\Omega \phi dP = \mu(X)$. In case this quantity is finite, then by Birkhoff's ergodic theorem, T is homogeneous and $(\Omega, a, P, \sigma, \phi)$ satisfies a strong law of large numbers. The main result of this section is:

THEOREM 4.1. *Let (X, \mathcal{B}, μ, T) be a c.e.m.p.t., $\Omega \in \mathcal{F}$ and $(\Omega, \mathcal{A}, P, \sigma, \phi)$ be the return time stochastic process of T on Ω . Then T is homogeneous (with return sequence $\{a_n\}$) iff $(\Omega, a, P, \sigma, \phi)$ satisfies a W.L.L.N. (with constants $\{b_n\}$). In this case, $\{a_n\}$ and $\{b_n\}$ are regularly varying with index 1, and*

$$(4.2) \quad \mu(\Omega) a_{[b_n]} \sim \frac{1}{\mu(\Omega)} b_{[a_n]} \sim n \quad \text{as } n \rightarrow \infty.$$

A function $a : \mathbf{R}_+ \rightarrow \mathbf{R}_+$ is termed *regularly varying with index $\alpha \in \mathbf{R}$* if

$$\frac{a(\lambda t)}{a(t)} \xrightarrow{t \rightarrow \infty} \lambda^\alpha \quad \forall \lambda > 0.$$

A sequence $\{a_n\}$ is so termed if the function $a_{[t]}$ has this property (where $[t]$ denotes the largest integer not greater than t). Regular variation is discussed in [5] and [10].

The proof of Theorem 4.1 will be in a sequence of lemmas. Firstly, let, for $x \in \Omega$,

$$\psi_n(x) = \sum_{k=1}^n 1_\Omega(T^k x) \quad \text{and} \quad \phi_n(x) = \sum_{k=0}^{n-1} \phi(\sigma^k x).$$

It follows that

$$(4.3) \quad \psi_{\phi_n(x)}(x) = n \quad \text{and} \quad \phi_{\psi_n(x)}(x) \leq n < \phi_{\psi_n(x)+1}(x).$$

We will assume that $\mu(\Omega) = 1$. No generality is lost, because we can normalise the measure μ , and (4.2) will follow from the regular variations of a_n and b_n .

LEMMA 4.2. *T is homogeneous with return sequence $\{a_n\}$ iff*

$$(4.4) \quad \frac{\psi_n}{a_n} \xrightarrow{P} 1.$$

PROOF. If T is homogeneous, then (4.4) is evident. Conversely, we have that $\forall n_k \rightarrow \infty, \exists m_l = n_{k_l} \rightarrow \infty$ such that

$$(4.5) \quad \frac{\psi_{m_l}}{a_{m_l}} \rightarrow 1 \quad \text{a.e. on } \Omega.$$

The convergence set for (4.5) is T -invariant, and so, containing Ω , must be (almost all of) X . Homogeneity now follows from the Hopf ergodic theorem. \square

LEMMA 4.3. *Let $a : \mathbf{R}_+ \rightarrow \mathbf{R}_+$ be monotone. If $\forall \varepsilon > 0, m \in \mathbf{N}$,*

$$(4.6) \quad a((1 - \varepsilon)mt) \leq ma(t) \leq a((1 + \varepsilon)mt) \quad \text{for } t \text{ large,}$$

then $a(t)$ is regularly varying with index 1.

PROOF. A little manipulation shows that (4.6) is true for m a positive rational number. Now suppose that λ, ε are positive rational numbers. We have

$$\lambda a(t) = (1 + \varepsilon) \left(\frac{\lambda}{1 + \varepsilon} \right) a(t) \leq (1 + \varepsilon) a \left((1 + \varepsilon) \left(\frac{\lambda}{1 + \varepsilon} \right) t \right) = (1 + \varepsilon) a(\lambda t)$$

for t large and similarly that

$$\lambda a(t) \geq (1 - \varepsilon) a(\lambda t) \quad \text{for } t \text{ large.}$$

In other words

$$\frac{a(\lambda t)}{a(t)} \xrightarrow[t \rightarrow \infty]{} \lambda \quad \text{for } \lambda > 0, \lambda \in \mathbf{Q}.$$

The required regular variation of $a(t)$ now follows from its monotonicity. \square

LEMMA 4.4. *Let $b_n \sim b_{n+1} \rightarrow \infty$ and $b_{mn} \sim mb_n$ as $n \rightarrow \infty, \forall m \in \mathbf{N}$. Then $a(t) = \min\{n \geq 1 : b_n \geq t\}$ is regularly varying with index 1, and $a_{b(t)} \sim t$ as $t \rightarrow \infty$.*

PROOF. We will show (4.6). It follows easily from the assumptions that:

- (i) $b_{[n/m]} \sim (1/m) b_n$ as $n \rightarrow \infty$, $\forall m \in \mathbb{N}$,
- (ii) $b_{a(t)} \geq t > b_{a(t)-1}$, $\forall t \geq 0$ where $b_0 = 0$,
- (iii) $b_{a(t)} \sim t$ as $t \rightarrow \infty$.

Using these, we can see that $\forall \varepsilon > 0$, $m \in \mathbb{N}$, $\exists t_0$ such that $\forall t \geq t_0$

$$b_{ma(t)} \geq (1 - \varepsilon) m b_{a(t)} \geq (1 - \varepsilon) m t$$

and

$$b_{[a((1+\varepsilon)mt)/m]} \geq \frac{1}{(1+\varepsilon)m} b_{a((1+\varepsilon)mt)} \geq t.$$

These inequalities yield (4.6) immediately. \square

LEMMA 4.5. If $\phi_n/b_n \xrightarrow{p} 1$ then T is homogeneous with return sequence $a(n)$, where $a(t) = \min \{n \geq 1 : b_n \geq t\}$ is regularly varying with index 1, and $b_{a(t)} \sim t$ as $t \rightarrow \infty$.

PROOF. We first establish the regular variation of $a(t)$, and the fact that $b_{a(t)} \sim t$ as $t \rightarrow \infty$. To do this, in view of the preceding lemma, it is sufficient to show that $b_n \sim b_{n+1}$ and $b_{mn} \sim m b_n$ as $n \rightarrow \infty$, $\forall m \in \mathbb{N}$. The first of these is clear. To see the second, let $\varepsilon > 0$ and $m \in \mathbb{N}$. Since

$$\frac{\phi_n}{b_n} \xrightarrow{p} 1$$

there is an n_0 such that $\forall n \geq n_0$

$$P\left(\bigcap_{k=0}^{m-1} \sigma^{-kn} \left[\left| \frac{\phi_n}{b_n} - 1 \right| \leq \varepsilon \right] \cap \left[\left| \frac{\phi_{mn}}{b_{mn}} - 1 \right| \leq \varepsilon \right]\right) > 0.$$

That is, $\forall n \geq n_0$, $\exists x \in \Omega$ such that

$$(1 - \varepsilon) b_n \leq \phi_n(\sigma^{kn} x) \leq (1 + \varepsilon) b_n \quad \text{for } 0 \leq k \leq m-1$$

and

$$(1 - \varepsilon) b_{mn} \leq \phi_{mn}(x) \leq (1 + \varepsilon) b_{mn}.$$

These, together with the fact that

$$\phi_{mn}(x) = \sum_{k=0}^{m-1} \phi_n(\sigma^{kn} x)$$

show that

$$\left| \frac{b_{mn}}{b_n} - m \right| \leq \frac{2\varepsilon}{1-\varepsilon} \quad \text{for } n \geq n_0.$$

To complete the proof of this lemma, it is now sufficient to show that

$$(4.7) \quad \forall \varepsilon > 0, \quad P(a((1-\varepsilon)n) \leq \psi_n \leq a((1+\varepsilon)n)) \rightarrow 1$$

(this, because of the now established regular variation of $a(t)$, and Lemma 4.2).

In view of (4.3), we have

$$\psi_n \geq a((1+\varepsilon)n) \Rightarrow \phi_{a((1+\varepsilon)n)} \leq n \leq \frac{1}{1+\varepsilon} b_{a((1+\varepsilon)n)},$$

$$\psi_n < a((1-\varepsilon)n) \Rightarrow \phi_{a((1-\varepsilon)n)} \geq n \geq \frac{1}{1-\varepsilon} b_{a((1-\varepsilon)n)}.$$

The assumption $\phi_n/b_n \xrightarrow{p} 1$ thus establishes (4.7). \square

LEMMA 4.6. Suppose T is homogeneous with return sequence $\{a_n\}$. Then $b(t) = \inf\{n \geq 1 : a_n \geq t\}$ is regularly varying with index 1, $a_{b(t)} \sim t$ as $t \rightarrow \infty$, and

$$\frac{\phi_n}{b(n)} \xrightarrow{p} 1.$$

PROOF. It is evident that:

- (i) $a_n \sim a_{n+1}$ as $n \rightarrow \infty$,
- (ii) $a_{b(t)} \geq t > a_{b(t)-1}$ for $t \geq 0$ where $a_0 = 0$,
- (iii) $a_{b(t)} \sim t$ as $t \uparrow \infty$.

Thus, from (4.3), we have, $\forall \varepsilon > 0$, that for t large

$$\phi_{[t]} > b((1+\varepsilon)t) \Rightarrow \psi_{b((1+\varepsilon)t)} \leq [t] \leq \frac{1}{1+\varepsilon} a_{b((1+\varepsilon)t)},$$

$$\phi_{[t]} \leq b((1-\varepsilon)t) \Rightarrow \psi_{b((1-\varepsilon)t)} \geq [t] \geq \frac{1-\varepsilon/2}{1-\varepsilon} a_{b((1-\varepsilon)t)},$$

whence, by Lemma 4.2, $\forall \varepsilon > 0$,

$$(4.8) \quad P(b((1-\varepsilon)t) \leq \phi_{[t]} \leq b((1+\varepsilon)t)) \xrightarrow{t \rightarrow \infty} 1.$$

To prove the lemma, it is sufficient now to show the regular variation of $b(t)$. As in the proof of Lemma 4.5, we deduce from (4.8) that $\forall \varepsilon > 0$, $m \in \mathbf{N}$, $\exists t_0$ such that $\forall t \geq t_0$, $\exists x \in \Omega$ such that

$$b((1-\varepsilon)t) \leq \phi_{[t]}(\sigma^{k[t]}x) \leq b((1+\varepsilon)t) \quad \text{for } 0 \leq k \leq m-1$$

and

$$b((1 - \varepsilon)mt) \leq \phi_{m[t]}(x) \leq b((1 + \varepsilon)mt).$$

Since

$$\phi_{m[t]}(x) = \sum_{k=0}^{m-1} \phi_{[t]}(\sigma^{k[t]}x),$$

we have established (4.6) and, hence, the regular variation of $b(t)$. \square

Lemmas 4.5 and 4.6 constitute the proof of Theorem 4.1, as the regular variations of the sequences $\{a_n\}$ and $\{b_n\}$ of Lemmas 4.6 and 4.5, respectively, are established by Lemmas 4.5 and 4.6, respectively.

We note that if T is both homogeneous, and weakly rationally ergodic (see [2]) then T satisfies a "strong L^1 ergodic theorem" on certain sets. Namely, if $A \in \mathcal{R}(T)$ and $B \subseteq A$ then

$$\int_A \left| \frac{1}{a_n(T)} \sum_{k=0}^{n-1} 1_B \circ T^k - \mu(B) \right| d\mu \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

This is because

$$\frac{1}{a_n(T)} \sum_{k=0}^{n-1} 1_B \circ T^k \rightarrow \mu(B)$$

both in measure (by homogeneity) and weakly in $L^1(A)$ (by weak rational ergodicity) and hence, strongly in $L^1(A)$.

THEOREM 4.2. *If (X, \mathcal{B}, μ, T) is a homogeneous m.p.t. and $\mu(X) = \infty$ then:*

$$\lim_{n \rightarrow \infty} \frac{1}{a_n(T)} \sum_{k=0}^{n-1} f \circ T^k = 0 \quad \text{a.e.} \quad \forall f \in L^1_+.$$

PROOF. (This is an adaptation of the proof of theorem 2 of [1].)

Suppose otherwise, then $\exists \alpha > 0$ ($\alpha \leq 1$) such that

$$\lim_{n \rightarrow \infty} \frac{1}{a(n)} \sum_{k=0}^{n-1} 1_A \circ T^k = \alpha \mu(A) \quad \text{a.e.} \quad \forall A \in \mathcal{B} \quad (a(n) = a_n(T)).$$

Fix $A \in \mathcal{B}$, $\mu(A) = 2$. Then $\exists \Omega \subseteq A$, $\mu(\Omega) = 1$ such that

$$(4.9) \quad \frac{1}{a(n)} \sum_{k=0}^{n-1} 1_A(T^k x) \geq \delta > 0, \quad \forall x \in \Omega, \quad n \geq 1.$$

We now restrict attention to $(\Omega, \mathcal{A}, p, \sigma, \phi)$ — the return time stochastic process of T on Ω . We will show $\int_{\Omega} \phi dP < \infty$, using, wherever possible, the proof of theorem 2 of [1].

Let $c(t) = P(\phi \geq t)$ and $L(t) = \int_0^t c(s)ds$. We have by Theorem 4.1 that $\phi_n/b(n) \xrightarrow{P} 1$, where $b(n)$ is regularly varying with index 1 and $b(a(n)) \sim a(b(n)) \sim n$ as $n \rightarrow \infty$.

By step 2 to the proof of theorem 2 of [1], we have

$$(4.10) \quad \sum_{n=1}^{\infty} c(b(n)) < \infty.$$

Let $\hat{b}(n) = \sum_{k=0}^{n-1} L(b(k))$. If we show that

$$(4.11) \quad \lim_{n \rightarrow \infty} \frac{\hat{b}(n)}{b(n)} \geq 1$$

then (in view of the regular variation of $b(n)$) we have, by step 4 of the above-mentioned proof, that

$$\sum_{n=1}^{\infty} c(\hat{b}(n)) < \infty$$

and hence that $\int_{\Omega} \phi dP < \infty$. We establish (4.11).

Let

$$f_n(x) = \begin{cases} \phi(x) & \text{if } \phi(x) \leq b(n), \\ 0 & \text{else.} \end{cases}$$

By (4.10), we have that for P -a.e. $x \in \Omega$

$$\phi(\sigma^n x) \neq f_n(\sigma^n x) \quad \text{for only finitely many } n \geq 1.$$

In particular

$$\frac{1}{b(n)} \sum_{k=0}^{n-1} f_k \circ \sigma^k \xrightarrow{P} 1.$$

Choosing a.e.-convergent subsequences, and using Fatou's lemma, we see that

$$\lim_{n \rightarrow \infty} \frac{1}{b(n)} \sum_{k=0}^{n-1} \int_{\Omega} f_k dP \geq 1.$$

This establishes (4.11) because

$$\int_{\Omega} f_k dP \leq L(b(k)). \quad \square$$

COROLLARY. *If $(\Omega, a, P, \sigma, \phi)$ is an (ergodic, \mathbb{N} -valued) stochastic process, and $\phi_n/b_n \xrightarrow{P} 1$, then*

$$E(\phi) = \infty \Rightarrow \overline{\lim}_{n \rightarrow \infty} \frac{\phi_n}{b_n} = \infty \quad \text{a.e.}$$

It follows from the remarks following corollary 3.5 of [3] that no odd restriction (of an inner function of \mathbb{R}^{2+}) preserving an infinite measure can be homogeneous.

The rest of this section is devoted to classifying the homogeneous Markov shifts. The following theorem of W. Feller will be used:

THEOREM 4.3 ([5] p. 236). *Let $\{X_n\}_{n=1}^\infty$ be independent, identically distributed random variables (i.i.d.r.v.s) with $X_k \geq 0$. Let $c(t) = P(X_1 \geq t)$ and $L(t) = \int_0^t c(s)ds$. Then $\exists b(n)$ such that $(1/b(n)) \sum_{k=1}^n X_k \rightarrow 1$ in probability iff $L(t)$ is slowly varying (i.e. regularly varying with index 0) and in this case*

$$b(n) \sim nL(b(n)) \quad \text{as } n \rightarrow \infty.$$

Let (X, \mathcal{B}, μ, T) be a c.e.m.p.t. We recall from [2] that a set $\Omega \in \mathcal{F}$ was said to be a *recurrent event* for T if $\forall 0 = n_0 \leq n_1 \leq n_2 \leq \dots \leq n_k$

$$\mu \left(\bigcap_{j=1}^k T^{-n_j} \Omega \right) = \mu(\Omega) \prod_{j=1}^k u_{n_j - n_{j-1}}$$

where $u_n = \mu(\Omega \cap T^{-n} \Omega) / \mu(\Omega)$.

The collection of recurrent events for T is denoted by $M(T)$, and T is said to *admit recurrent events* if $M(T) \neq \emptyset$. It is not hard to see that $M(T) \neq \emptyset$ iff T has a Markov shift factor, and that if $(\Omega, \mathcal{A}, P, \sigma, \phi)$ is the return time stochastic process of T on $\Omega \in M(T)$ then $(\phi \circ \sigma^n)$ are i.i.d.r.v.s and

$$(4.12) \quad \sum_{n=0}^{\infty} u_n \lambda^n = ((1 - \lambda) \sum_{n=0}^{\infty} c_n \lambda^n)^{-1}$$

where $u_n = \mu(\Omega \cap T^{-n} \Omega) / \mu(\Omega)$ and $c_n = P(\phi > n)$.

It was shown in [2] that if $\exists \Omega \in M(T)$ then T is rationally ergodic (hence by Theorem 1.1 weakly homogeneous), and that $(1/\mu(\Omega)) \sum_{k=0}^n u_k$ is a return sequence for T .

Now suppose that T is a c.e.m.p.t. and that $\Omega \in M(T)$, $\mu(\Omega) = 1$. Equation (4.12) implies, through Karamata's Tauberian theorem (see [5] and [10], that $\sum_{k=0}^n u_k$ is regularly varying with index 1 iff $L(n) = \sum_{k=0}^n c_k$ is slowly varying.

Thus, if T is homogeneous, then $L(n)$ is slowly varying; and, by Feller's theorem, if $L(n)$ is slowly varying, then $(\Omega, \mathcal{A}, P, \sigma, \phi)$ satisfies a W.L.L.N.

We have proved

THEOREM 4.4. *If T is a c.e.m.p.t. and $M(T) \neq \emptyset$, then T is homogeneous iff $a_n(T)$ is regularly varying with index 1.*

REFERENCES

1. J. Aaronson, *On the ergodic theory of non-integrable functions and infinite measure spaces*, Israel J. Math. **27** (1977), 163–173.
2. J. Aaronson, *Rational ergodicity and a metric invariant for Markov shifts*, Israel J. Math. **27** (1977), 93–123.
3. J. Aaronson, *Ergodic theory for inner functions of the upper half plane*, Ann. Inst. H. Poincaré Sect. B **14** (1978), 233–255.
4. S. D. Chatterji, *A general strong law*, Invent. Math. **9** (1970), 235–245.
5. W. Feller, *Introduction to Probability Theory and its Applications*, Vol. 2, Wiley, New York, 1966.
6. A. Hajian, Y. Ito and S. Kakutani, *Invariant measures and orbits of dissipative transformations*, Advances in Math. **9** (1972), 52–66.
7. E. Hopf, *Ergodentheorie*, Chelsea, New York, 1948.
8. S. Kakutani, *Induced measure preserving transformations*, Proc. Imp. Acad. Sci. Tokyo **19** (1943), 635–641.
9. J. Komlos, *A generalisation of a problem of Steinhaus*, Acta Math. Acad. Sci. Hungar. **18** (1967), 217–229.
10. E. Seneta, *Regularly Varying Functions*, Springer Lecture Notes **508**, Berlin, 1976.

DEPARTMENT DE MATHÉMATIQUES ET INFORMATIQUE
UNIVERSITÉ DE RENNES
AVENUE GEN. LECLERC
35042 RENNES, FRANCE